NUM. V.

Ejusdem

De Integratione
Æquationum Differentialium altiorum graduum.

§. I.

QUanquam ad reſolvendas æquationes differentiales primi gradus plurimæ adhuc excogitatæ ſunt methodi, atque in hoc negotio ſummi Geometræ operam ac ſtudium collocaverunt; tamen parum etiam nunc ſubſidii attulerunt ad æquationes differentiales altiorum graduum tractandas, easque vel conſtruendas vel integrandas. Æquationes quidem differentiales ſecundi gradus ita reſolvi ſolent, ut per idoneam ſubſtitutionem ad æquationes primi gradus reducantur, quo facto earum reſolutio ad viam magis tritam & cognitam revocatur: atque in hoc negotio nonnulla ſubſidia ante complures annos excogitavi, quorum ope innumerabiles æquationes differentiales ſecundi gradus ad primum gradum deprimi, atque adeo ſive conſtrui ſive integrari poſſunt. At vero in æquationibus differentialibus tertii, altiorisve gradus, ſimilia artificia, quibus eæ ad gradum inferiorem traduci queant, parum ac plerumque nihil proſunt, cum hoc pacto ad æquationes differentiales ſecundi vel etiam altioris gradus tam complicatas perveniatur, ut ulterius

 tra-

tractari omnino nequeant. Quamobrem in hoc negotio non parum utilitatis afferet methodus, quam hic ſum expoſiturus, cujus beneficio plurimæ æquationes differentiales altiorum graduum ſine prævia reductione ad gradus inferiores ſtatim integrari, atque æquationes integrales in terminis finitis exhiberi poſſunt.

§. II.

Sint y & x variabiles, quibus æquatio differentialis cujuscunque gradus contineatur, in qua differentiale dx aſſumtum ſit conſtans: atque habeat altera variabilis y cum ſuis differentialibus dy, ddy, d^3y, &c. in ſingulis terminis unicam dimenſionem, ita ut æquatio, cujuscunque demum ſit gradus, ſequentem induat formam:

$$0 = Ay + \frac{B\,dy}{dx} + \frac{C\,ddy}{dx^2} + \frac{D\,d^3y}{dx^3} + \frac{E\,d^4y}{dx^4} + \frac{F\,d^5y}{dx^5} + \&c.$$

in qua litteræ A, B, C, D, &c. ſignificent quantitates vel conſtantes, vel alteram variabilem x utcunque involventes. Manifeſtum autem eſt, hanc æquationem latiſſime patere, non ſolum enim ob coefficientes indeterminatos A, B, C, D, &c. quos ſimul functiones quascunque ipſius x aſſumimus, maxime eſt generalis, ſed etiam æquationes differentiales cujuscunque gradus in ſe complectitur. Hanc igitur æquationem, quibus caſibus integrationem admittat, in hac diſſertatione evolvam.

§. III.

Primum quidem perſpicuum eſt, æquationem integralem completam præter quantitates conſtantes in ipſa æquatione

tione differentiali contentas, tot novas conſtantes arbitrarias in ſe complecti oportere, quoti fuerit gradus æquatio differentialis propoſita. Quodſi enim ponamus, eam æquationem eſſe differentialem gradus n, ita ut ultimus illius terminus ſit $\frac{N\,d^n y}{dx^n}$; per unam integrationem ea reducetur ad gradum $n - 1$. per duas integrationes ſucceſſive inſtitutas ad gradum $n - 2$; per tres ad gradum $n - 3$; & ita porro. Ex quo intelligitur, demum poſt n integrationes ad æquationem integralem terminis finitis expreſſam perveniri Quoniam vero per unamquamque integrationem una conſtans arbitraria in integrale ingreditur, manifeſtum eſt, integrale completum n conſtantes arbitrarias complecti oportere.

§. IV.

Æquatio igitur integralis completa tot conſtantes arbitrarias in ſe complectitur, quot exponens n continebit unitates; hæcque æquatio integralis æque late patere cenſenda eſt, atque ipſa æquatio differentialis gradus n: ita ut nullus valor finitus pro y aſſumtus æquationi differentiali ſatisfacere queat, qui non contineatur in æquatione integrali completa. Quodſi autem in iſta æquatione integrali completa una pluresve illarum conſtantium arbitrariarum pro lubitu determinentur, tum habebitur quidem æquatio quæſito ſatisfaciens, at ea non amplius erit completa, ſed tantum æquatio integralis particularis, quæ non omnes poſſibiles valores ipſius y æquationi differentiali ſatisfacientes in ſe complectitur. Probe igitur diſcerni oportet æquationem integralem

completam a particulari; atque ſi æquationi differentiali perfecte ſatisfacere velimus, æquationem integralem completam inveniri oportet.

§. V.

Ad cognoſcendum autem, utrum æquatio integralis exhibita ſit completa, nec ne, criterium ex allatis facile colligitur. Primum enim ea æquationi propoſitæ differentiali ſatisfacere debet, quod fit, dum facta ſubſtitutione æquatio identica reſultat; alioquin enim illa æquatio nequidem foret integralis. Præterea vero neceſſe eſt, ut æquatio integralis tot contineat quantitates conſtantes arbitrarias, quoti fuerit gradus æquatio differentialis propoſita. Si enim pauciores in ea inſint conſtantes, tum æquatio integralis data non erit completa, ſed tantum particularis. In enumeratione autem conſtantium arbitrariarum probe cavendum eſt, ne per numerum diverſarum litterarum fallamur, neque pro diverſis quantitatibus habeamus, quæ per ſe invicem determinantur.

§. VI.

Quo diſcrimen inter æquationes integrales completas & incompletas clarius intelligatur, juvabit rem exemplo illuſtraſſe. Sit igitur propoſita hæc æquatio differentialis $aady + yydx = (aa + xx)dx$; cui ſatisfacere patet hunc valorem $y = x$, quippe qui ſubſtitutus producit æquationem identicam. Eſt igitur $y = x$ æquatio integralis, at minime completa, cum ea neque conſtantem a, quæ in æquatione differentiali ineſt, neque præterea aliam conſtantem arbitra-

bitrariam contineat, quemadmodum æquatio differentialis primi gradus poſtulat. Vehementer igitur falleretur, qui hanc æquationem $y = x$ pro integrali completa hujus $aady + yydx = (aa + xx)dx$ venditare vellet: æquatio enim integralis completa eſt

$$y = x + \frac{aabe^{\frac{-xx}{aa}}}{aa + b\int e^{\frac{-xx}{aa}}dx}$$: quæ faciendo conſtantem arbitariam $b = o$ utique reddit integralem particularem $y = x$.

§. VII.

Simili modo videmus huic æquationi differentio-differentiali $y = \frac{xdy}{dx} + \frac{axddy}{dx^2}$ ſatisfacere hanc æquationem finitam $y = x$; procul autem abeſt, quominus ſit integralis completa, omnemque vim æquationis differentio-differentialis exhauriat, quoniam æquatio integralis completa præter conſtantem a duas quantitates arbitrarias continere debet. Videmus vero etiam hanc æquationem $y = nx$ ſatisfacere, quæ autem, quia unicam conſtantem n continet, tantum modo eſt particularis. Æquatio autem integralis completa eſt

$$y = nx + bx\int \frac{e^{\frac{-x}{a}}dx}{xx}$$, quæ præter conſtantem a duas continet conſtantes arbitrarias b & n; uti natura rei poſtulat.

§. VIII.

§. VIII.

Cum autem omnes æquationes integrales particulares in completa contineantur, patet ex pluribus integralibus particularibus completam conſtare; atque adeo ex integralibus particularibus integrale completum conflabitur. Sæpe numero quidem æque difficile eſt ex cognitis aliquot integralibus particularibus integrale completum vel ſaltem integrale latius patens, atque idem ex ipſa æquatione differentiali per integrationem colligere. At vero æquatio quam tractare ſuscepimus

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \&c.$$

ita eſt comparata, ut cognitis valoribus particularibus ipſius y duobus pluribusve, ex iis facile valor latius patens illos nempe valores in ſe complectens ipſius y formari queat. Hocque pacto ex ſufficienti numero valorum particularium pro y inventorum valor completus, ſeu æquatio integralis completa concinnari poterit.

§. IX.

Primo autem perſpicitur ſi p fuerit valor conveniens ipſius y, ita ut ſit $y = p$, tum etiam fore $y = \alpha p$; namque ſi valor p loco y ſubſtitutus reddit $Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2}$ $+ \&c. = 0$, tum valor αp loco y ſubſtitutus eandem expreſſionem evaneſcentem efficiet: hocque modo una conſtans arbitraria α in æquationem integralem particularem

$y = p$

$y = p$ introduci poteſt. Sin autem præterea hæc æquatio $y = q$ ſatisfaciat propoſitæ, tum pari modo quoque ſatisfaciet $y = \beta q$: ex his autem duobus valoribus particularibus $y = \alpha p$ & $y = \beta q$, colligetur hic latius patens $y = \alpha p + \beta q$. Si enim expreſſio $Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx} + \frac{Dd^3y}{dx^3} +$ &c. nihilo æqualis redditur, poſito tam αp quam βq loco y, manifeſtum eſt, eandem expreſſionem nihilo æqualem fieri debere, ſi loco y ponatur $\alpha p + \beta q$.

§. X.

Simili modo ſi p, q, r, s, &c. fuerint ejusmodi functionis ipſius x, ut ſingulæ ſeorſim loco y ſubſtitutæ expreſſionem $Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} +$ &c. evaneſcentem efficiant, tum etiam hic valor $\alpha p + \beta q + \gamma r + \delta s +$ &c. loco y ſubſtitutus eandem expreſſionem nihilo æqualem producet. Quare ſi p, q, r, s, &c. fuerint valores particulares ipſius y, qui ipſi ex æquatione propoſita conveniunt, tum ex iis colligitur iſte valor longe latius patens $y = \alpha p + \beta q + \gamma r + \delta s +$ &c. æquationi propoſitæ pariter ſatisfaciens. Hicque valor adeo erit completus, ſi tot affuerint conſtantes arbitrariæ α, β, γ, δ, &c. quoti fuerit gradus æquatio differentialis propoſita. Facilem igitur nacti ſumus modum, ex pluribus valoribus particularibus ipſius y, ejus valorem completum aſſignandi, qui omnes omnino valores ipſius y æquationi ſatisfacientes in ſe complectatur: ſicque

habebi-

habebitur æquatio integralis in terminis finitis completa.

§. XI.

Totum ergo negotium pro inveniendo integrali completo æquationis propoſitæ

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + \frac{Nd^ny}{dx^n}$$

huc reducitur, ut valores particulares inveſtigemus, qui pro y ſubſtituti æquationem identicam reddant. Tot autem ejusmodi valoribus particularibus erit opus, quoad iis præſcripto modo colligendis tot conſtantes arbitrariæ affuerint, quot exponens maximus n continet unitates. Quare ſi ſingulæ æquationes particulares unam ſecum gerant conſtantem arbitrariam, ejusmodi æquationes numero n requiruntur, ad æquationem integralem completam conſtituendam. Sin autem quædam harum æquationum particularium plures una conſtantes arbitrarias implicent, tum eo paucioribus opus erit æquationibus particularibus ad completam ex iis colligendam.

§. XII.

Denotent jam omnes litteræ A, B, C, D, &c. quantitates conſtantes, ita ut integrari debeat hæc æquatio differentialis gradus n

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + \frac{Nd^ny}{dx^n}.$$

Quoniam y cum ſuis differentialibus ubique unicam dimenſionem complet, ſecundum methodum meam in Tomo III.

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Comment. Academiæ Petropol. traditam hæc æquatio differentialis uno gradu deprimetur, ſi ponamus $y = e^{\int p dx}$: unde ſingula differentialia ipſius y erunt

$$\frac{dy}{dx} = e^{\int p dx} p$$

$$\frac{ddy}{dx^2} = e^{\int p dx} \left(pp + \frac{dp}{dx}\right)$$

$$\frac{d^3y}{dx^3} = e^{\int p dx} \left(p^3 + \frac{2p\,dp}{dx} + \frac{ddp}{dx^2}\right)$$

$$\frac{d^4y}{dx^4} = e^{\int p dx} \left(p^4 + \frac{6pp\,dp}{dx} + \frac{4p\,ddp}{dx^2} + \frac{3dp^2}{dx^2} + \frac{d^3p}{dx^3}\right)$$

&c.

qui valores ſi in propoſita ſubſtituantur, ea dividi poterit per $e^{\int p dx}$, atque remanebit æquatio differentialis gradus $n-1$.

§. XIII.

Hic autem primum patet, ſi capiatur p conſtans, ita ut ejus differentialia dp, ddp, d^3p, &c. evaneſcant, tum ob A, B, C, D, &c. conſtantes, variabilem x prorſus ex æquatione eſſe exceſſuram; atque in hac hypotheſi orietur ſequens æquatio algebraica:

$$0 = A + Bp + Cp^2 + Dp^3 + Ep^4 \ldots\ldots + Np^n:$$

ex qua ſi valor aliquis pro p eruatur, ſimul habebitur æquatio integralis particularis $y = e^{px}$, æquationi differentiali propoſitæ ſatisfaciens; ſatisfaciet ergo etiam uti vidimus hæc

æquatio $y = \alpha e^{px}$, quoties p fuerit quantitas conſtans ac radix hujus æquationis algebraicæ $0 = A + Bp + Cp^2 + Dp^3 + \ldots + Np^n$.

§. XIV.

Perduximus ergo inventionem valorum particularium pro variabili y, ad reſolutionem æquationis algebraicæ n dimenſionum, pro qua ſumamus hanc

$$0 = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$$

hujusque æquationis ſingulæ radices ſeu diviſores dabunt totidem valores particulares ipſius y. Si enim fuerit $pz - q$ diviſor iſtius æquationis, ex quo oritur $z = \frac{q}{p}$, erit $y = \alpha e^{\frac{qx}{p}}$; qui valor particularis unam continet conſtantem arbitrariam α. Cum autem illa æquatio algebraica n dimenſionum contineat n radices ſeu diviſores, hinc quoque orientur n valores particulares pro y; qui junctim ſumti dabunt valorem univerſalem pro y; hicque ſimul erit valor completus, eo quod n contineat conſtantes arbitrarias; quod eſt criterium æquationis integralis completæ.

§. XV.

Si ergo æquationis iſtius algebraicæ n dimenſionum omnes radices fuerint reales, tum prodibit valor completus pro y in terminis realibus expreſſus, eritque aggregatum n formularum exponentialium hujus ſpeciei $\alpha e^{qx:p}$: hocque

adeo

adeo caſu integrale completum per ſolam logarithmicam ſeu quadraturam hyperbolæ exprimi poterit. Quodſi autem aliquot radicum illius æquationis algebraicæ fuerint imaginariæ, tum in integrale completum ingredientur formulæ exponentiales imaginariæ; quas infra per quadraturam circuli conſtruere docebo. Præcipua difficultas in hoc verſatur, quando duæ pluresve radices æquationis ſint inter ſe æquales, tum enim ob plures formulas exponentiales æquales, numerus conſtantium arbitrariarum diminuitur; atque obhanc cauſam integrale inventum non amplius erit completum.

§. XVI.

Utrique incommodo medelam afferemus, ſi nexum inter æquationem differentialem propoſitam

$$0 = Ay + \frac{B\,dy}{dx} + \frac{C\,ddy}{dx^2} + \frac{D\,d^3y}{dx^3} + \ldots\ldots + \frac{N\,d^n}{dx^n}$$

atque inter æquationem algebraicam formatam

$$0 = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$$

attentius contemplemur. Quemadmodum enim ex hac illa oritur, ſi loco z^0 ponatur y, loco z vero $\frac{dy}{dx}$, & generaliter loco z^k ſcribatur $\frac{d^ky}{dx^k}$: ita ſimilimodo ex factoribus ſingulis æquationis algebraicæ formabuntur æquationes differentiales, quæ neceſſario in æquatione differentiali propoſita continebuntur, atque ex quibus proinde valores particulares pro y reperientur. Sic ſi $pz-q$ vel $q-pz$ fuerit diviſor æquationis algebraicæ, ex hoc per legem ne-

 xus

xus oritur hæc æquatio differentialis $qy - \frac{pdy}{dx} = 0$ quæ integrata dat $y = \alpha e^{\frac{qx}{p}}$, quæ eſt ea ipſa, quam ex eodem factore $pz - q$ elicuimus.

§. XVII.

Hinc intelligitur ſi habeatur diviſor quicunque æquationis illius algebraicæ puta, $p + qz + rzz$ tum æquationem ex hoc diviſore oriundam $py + \frac{qdy}{dx} + \frac{rddy}{dx^2} = 0$ dare valorem pro y, qui etiam ſatisfacit æquationi differentiali propoſitæ. Ex hoc ergo illam difficultatem tollere poterimus, quæ locum habet, ſi æquatio algebraica habeat duos pluresve factores æquales. Sit igitur $(p - qz)^2$ diviſor æquationis algebraicæ, ex hocque evoluto reſultabit hæc æquatio differentio-differentialis $ppy - \frac{2pqdy}{dx} + \frac{qqddy}{dx^2} = 0$. Ponamus $y = e^{\frac{px}{q}} u$, facta que ſubſtitutione habebimus $ddu = 0$, hincque $u = \alpha + \beta x$. Quare ex factore quadrato $(p - qz)^2$ oritur ſequens valor $y = e^{\frac{px}{q}}(\alpha + \beta x)$, qui duas conſtantes arbitrarias complectitur.

§. XVIII.

Si æquatio algebraica habeat diviſorem cubicum $(p - qz)^3$, tum in æquatione differentiali propoſita continebitur hacc

$p^3 y -$

$p^3 y - \frac{3ppq\,dy}{dx} + \frac{3pqq\,ddy}{dx^2} - \frac{q^3 d^3 y}{dx^3} = 0$ quæ poſito $y = e^{\frac{px}{q}} u$, transmutabitur in hanc: $d^3 u = 0$; unde oritur $u = \alpha + \beta x + \gamma xx$, ex quo æquationi propoſitæ ſatisfaciet iſte valor particularis $y = e^{\frac{px}{q}} (\alpha + \beta x + \gamma xx)$. Simili modo ſi æquatio algebraica $0 = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$ habeat diviſorem biquadratum $(p - qz)^4$, tum ex eo naſcetur iſta æquatio particularis ſatisfaciens $y = e^{\frac{px}{q}} (\alpha + \beta x + \gamma xx + \delta x^3)$. Atque generaliter ſi diviſor ſit $(p - qz)^k$, erit valor inde ortus $y = e^{\frac{px}{q}} (\alpha + \beta x + \gamma xx + \delta x^3 + \ldots \ldots . kx^{k-1})$. ita ut is k conſtantes imaginarias involvat.

§. XIX.

Si cui dubium ſuperſit, an hoc modo ex diviſoribus compoſitis, in quibus z plures una habet dimenſiones, æquationis $0 = A + Bz + Cz^2 + Dz^3 + \ldots \ldots + Nz^n$ recte deducantur valores pro y, qui æquationi propoſitæ:

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots \ldots + \frac{Nd^ny}{dx^n}$$

hoc dubium ex natura rei facile tolli poterit. Sit diviſor utcunque compoſitus $p + qz + rzz + sz^3 + \&c.$ ex eoque formetur æquatio $0 = py + \frac{qdy}{dx} + \frac{rddy}{dx^2} + \frac{sd^3y}{dx^3} + \&c.$

$+$ &c. atque patebit valorem completum ipſius y pro hac æquatione prodire, ſi omnes valores ipſius y quos diviſores ſimplices æquationis $0 = p + qz + rzz + sz^3 +$ &c. ſuppeditant, in unam ſummam colligantur; at diviſores ſimplices hujus æquationis ſimul ſunt diviſores ſimplices illius $0 = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$; & hanc obrem valor ipſius y ex illa factore compoſito ortus, ſimul eſt valor conveniens æquationis propoſitæ differentialis

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + \frac{Nd^ny}{dx^n};$$

§. XX.

Inventis autem valoribus ipſius y, qui ex aliquot diviſoribus ſimplicibus inter ſe æqualibus æquationis $0 = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$ oriuntur, altera difficultas ſolvenda reſtat, ſi hæc æquatio habeat radices imaginarias. Conſtat autem, ſi quæpiam æquatio habeat radices imaginarias, earum numerum ſemper eſſe parem; atque ego alibi oſtendi has radices imaginarias perpetuo binis conjungendis in ejusmodi paria dispesci poſſe, quarum tam ſumma quam productum fiat quantitas realis. Hinc loco diviſorum imaginariorum prodibunt diviſores compoſiti duarum dimenſionum hujus formæ $p - qz + rzz$ reales, qui autem diviſores ſimplices habeant imaginarios. Erit ergo in tali diviſore compoſito $qq < 4pr$; unde $\frac{q}{2\sqrt{pr}} < 1$.

$\prec$ 1. Poſito ergo ſinu toto $= 1$ erit $\frac{q}{2\sqrt{pr}}$ coſinus cujuspiam anguli realis, qui ſit $= \Phi$; fietque $q = 2\sqrt{pr}.\cos A.\Phi$, ex quo generalis forma diviſorum compoſitorum, qui diviſores imaginarios in ſe contineant, erit hujusmodi $p - 2z\sqrt{pr}.\cos A\Phi + rzz$.

§. XXI.

Sit igitur æquationis $o = A + Bz + Cz^2 + \&c.$ ejusmodi diviſor $p - 2z\sqrt{pr}.\cos A\Phi + rzz$; ex quo inveniri debeat conveniens valor ipſius y. At ex hoc diviſore naſcitur iſta æquatio differentio - differentialis

$o = py - \frac{2dy\sqrt{pr}}{dx}\cos A.\Phi + \frac{rddy}{dx}$ ad quam integrandam ponatur $y = e^{fx\cos A.\Phi} u$ poſito brevitatis gratia $f = \sqrt{\frac{p}{r}}$ fietque $ffudx^3 (\sin A\Phi)^2 + ddu = o$. Multiplicetur per $2du$ & integretur erit $ffuudx^2 (\sin A\Phi)^2 + du^2 = \alpha^2 ffdx^2 (\sin A\Phi)^2$, unde $fdx\sin A\Phi = \frac{du}{\sqrt{(\alpha^2 - u^2)}}$; quæ integrata dat $fx\sin A\Phi + \beta = A\sin.\frac{u}{\alpha}$. Ex qua æquatione fit $u = \alpha\sin A.(fx\sin A.\Phi + \beta)$. Conſequenter habetur $y = \alpha e^{fx\cos A\Phi}\sin A(fx\sin A\Phi + \beta)$ qui erit valor conveniens ipſius y pro æquatione propoſita.

§. XXII.

§. XXII.

Eadem vel æquivalens expreſſio pro y colligitur ex factoribus ſimplicibus etſi imaginariis æquationis $0 = p - 2z\sqrt{pr}.\cos A\Phi + rzz$: quæ poſito $f = \sqrt{\frac{p}{r}}$ abit in hanc $0 = ff - 2fz\cos A\Phi + zz$, cujus radices ſunt $z = f\cos A\Phi \pm f\sqrt{-1}.\sin A\Phi$. Hinc pro y reſultant valores $e^{fx\cos A\Phi + fx\sqrt{-1}.\sin A\Phi}$ & $e^{fx\cos A\Phi - fx\sqrt{-1}.\sin A\Phi}$ quibus conjunctis fit $y = e^{fx\cos A\Phi}(\eta e^{+fx\sqrt{-1}.\sin A\Phi} + \theta e^{-fx\sqrt{-1}.\sin A\Phi})$

His autem exponentialibus in ſeries converſis prodibit

$$y = e^{fx\cos A\Phi}\begin{cases}(\eta+\theta)\left(1 - \frac{ffxx(\sin A\Phi)^2}{1.\ 2} + \frac{f^4x^4\ (\sin A\Phi)^4}{1.\ 2.\ 3.\ 4} + \&c.\right)\\ (\eta-\theta)\sqrt{-1}\left(fx\sin A\Phi - \frac{f^3x^3\ (\sin A\Phi)^3}{1.\ 2.\ 3} + \ \&c.\ \right)\end{cases}$$

Poſito ergo $\eta + \theta = \alpha$ & $(\eta - \theta)\sqrt{-1} = \beta$, orietur ſummatis his ſeriebus infinitis $y = e^{fx\cos A\Phi}(\alpha\cos A.fx\sin A\Phi + \beta\sin A.fx\sin A\Phi)$: quæ expreſſio ad priorem facile reducitur.

§. XXIII.

Hinc adipiſcimur modum inveniendi valorem ipſius y, ſi duo pluresve hujusmodi diviſores compoſiti fuerint inter ſe æquales. Sit enim $(ff - 2fz\cos A\Phi + zz)^2$ diviſor æquationis algebraicæ: quoniam is reducitur ad $(z - f\cos A\Phi - f\sqrt{-1}.\sin A\Phi)^2\ (z - f\cos A\Phi + f\sqrt{-1}.\sin A\Phi)^2$,

erit

erit per præcedentia valor ipſius y hinc oriundus: $y =$
$$e^{fx\cos A\Phi + fx\sqrt{-1}.\sin A\Phi}(\eta+\theta x) + e^{fx\cos A\Phi - fx\sqrt{-1}\sin A\Phi}(\iota+\kappa x)$$

Cum autem ſit $e^{+fx\sqrt{-1}.\sin A\Phi}\eta + e^{-fx\sqrt{-1}.\sin A\Phi}\iota = \alpha \cos A.fx\sin A\Phi + \beta \sin A.fx\sin A\Phi$, hinc colligitur fore $y =$
$$e^{fx\cos A\Phi}\left((\alpha+\beta x)\cos A.fx\sin A\Phi + (\gamma+\delta x)\sin A.fx\sin A\Phi\right)$$

§. XXIV.

Quod ſi autem cubus aliave poteſtas ipſius $ff - 2fz \cos A\Phi + zz$ fuerit diviſor æquationis algebraicæ $0 = A + Bz + Czz + Dz^3 + \ldots\ldots + Nz^n$, tum ex poteſtatibus iisdem factorum ſimplicium imaginariorum valores ipſius y eruantur ſecundum §. 18. & in unam ſummam conjiciantur. Quo facto quantitates exponentiales imaginariæ in ſinus & coſinus arcuum circularium converti poſſunt ope hujus lemmatis
$$e^{+fx\sqrt{-1}.\sin A\Phi}\eta x^k + e^{-fx\sqrt{-1}.\sin A\Phi}\theta x^k = \alpha x^k \cos A.fx\sin A\Phi + \beta x^k \sin A.fx\sin A\Phi.$$
Sic ſi $(ff - 2fz\cos A\Phi + zz)^4$ fuerit diviſor æquationis algebraicæ, tum ex eo ſequens naſcetur æquatio integralis $y = e^{fx\cos A\Phi}((\alpha + \beta x + \gamma x^2 + \delta x^3) \cos A.fx\sin A\Phi + (\varepsilon + \zeta x + \eta x^2 + \theta x^3) \sin A fx\sin A\Phi)$.

§. XXV.

Expreſſiones iſtæ pluribus modis immutari poſſunt, pro-

 ut

ut conſtantes aliis atque aliis modis exprimantur. Commodiſſima autem videtur hæc transmutatio, qua valores ipſius y ad formam §. 21. inventam reducuntur. Ita hæc forma $\mu x^k \cos A . \int x \sin A\, \Phi + \nu x^k \sin A . \int x \sin A\, \Phi$ ſi ponatur $\mu = \lambda \sin A . p$, & $\nu = \lambda \cos A . p$, transmutabitur in hanc $\lambda x^k \sin A (\int x \sin A\, \Phi + p)$. Qamobrem ex factore indefiniti exponentis $(ff - 2fz \cos A . \Phi + zz)^k$ formabitur ſequens valor ipſius y;

$$y = e^{\int x \cos A \Phi} (\alpha \sin A (\int x \sin A \Phi + \mathfrak{A}) + \beta x \sin A (\int x \sin A \Phi + \mathfrak{B})$$
$$+ \gamma x^2 \sin A (\int x \sin A \Phi + \mathfrak{C}) + \ldots\ldots + \varkappa x^{k-1} \sin A (\int x \sin A \Phi + \mathfrak{K}))$$

Hocque pacto ex omnibus diviſoribus, utcunque fuerint comparati, valores reales pro variabili y inveniuntur.

§. XXVI.

Quod jam ad conſtantes arbitrarias, quæ in valores ipſius y hoc modo inveniendos ingrediuntur, attinet; patet primo, ex factoribus ſimplicibus formæ $f - z$, oriri valores ipſius y unicam conſtantem arbitrariam continentes: deinde valor ipſius y, qui oritur ex factore $(f - z)^k$, continet k conſtantes arbitrarias. Porro ex factore compoſito $ff - 2fz \cos A \Phi + zz$ prodit valor ipſius y duas conſtantes arbitrarias complectens: atque ex hujusmodi factorum poteſtate quacunque $(ff - 2fz \cos A . \Phi + zz)^k$ formatur valor ipſius y, in quo $2k$ conſtantes arbitrariæ inſunt; ita ut numerus conſtantium arbitrariarum æqualis ſit numero dimenſionum ipſius z, quas hæc variabilis in diviſore obtinet, ex quo valor ipſius y eſt erutus.

§. XXVII.

§. XXVII.

Quodſi ergo æquatio algebraica, quam ex æquatione differentiali propoſita formavimus

$$o = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots + Nz^n$$

in factores ſuos ſive ſimplices ſive compoſitos reales, ſive tales, qui ſint poteſtates ſimplicium compoſitorumve, reſolvatur, atque modo deſcripto ex ſingulis valores convenientes ipſius y formentur, tum hi omnes ipſius y valores junctim conſiderati tot continebunt conſtantes arbitrarias, quot in exponente n inſunt unitates. Omnes igitur iſti valores in unam ſummam collecti non ſolum valorem præbebunt pro y, qui æquationi propoſitæ

$$o = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots + \frac{Nd^ny}{dx^n}$$

ſatisfaciat, verum etiam iſte ipſius y erit valor completus, qui omnes poſſibiles valores huic æquationi convenientes in ſe complectetur. Hoc ergo pacto æquatio iſta differentialis perfecte integratur in terminis finitis, neque integrale unquam alias præter hyperbolæ atque circuli quadraturas requirit.

Problema. I.

§. XXVIII.

Si propoſita fuerit æquatio differentialis gradus n hujus formæ

$$o = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + \frac{Nd^ny}{dx^n}$$

in qua elementum dx poſitum eſt conſtans, ac litteræ A, B,

 C, D,

C, D . . . . . N denotant coefficientes conſtantes quoscunque: invenire hujus æquationis integrale in terminis finitis realibus.

Solutio.

Scribatur 1 loco y; z loco $\frac{dy}{dx}$; z^2 loco $\frac{ddy}{dx^2}$; & generaliter z^k loco $\frac{d^k y}{dx^k}$; hincque formetur ſequens æquatio algebraica n dimenſionum.

$$o = A + Bz + Cz^2 + Dz^3 + \dots\dots + Nz^n$$

Hujus æquationis porro quærantur omnes diviſores ſimplices reales, quos quidem involvit; &, ſi habeat diviſores imaginarios, eorum loco capiantur diviſores compoſiti reales, in quibus z habeat duas dimenſiones, quoniam ſemper bini factores imaginarii unum factorem compoſitum realem conſtituunt. Ex ſingulis diviſoribus deinceps formentur ſequenti modo valores particulares pro y. Ex factore ſcilicet quolibet ſimplici, qui alios non habet ſui æquales, hujus formæ $f - z$, oritur iſte valor $y = \alpha e^{fx}$. Ex duobus autem pluribusve factoribus æqualibus conjunctim ſumtis valores ipſius y determinari debent. Nempe ex factore $(f-z)^2$ oritur $y = (\alpha + \beta x) e^{fx}$: ex factore $(f-z)^3$ oritur $y = (\alpha + \beta x + \gamma xx) e^{fx}$; atque generaliter ex factore $(f-z)^k$ deducitur $y = e^{fx} (\alpha + \beta x + \gamma xx + \dots\dots + \varkappa x^{k-1})$.

Quod

Quod ad factores compoſitos attinet, ſi illa æquatio algebraica habeat factorem $ff - 2fz\cos A.\Phi + zz$, qui ſui ſimilem inter reliquos non habeat, erit valor ex eo oriundus $y = e^{fx\cos A\Phi}\,\alpha\sin A(fx\sin A\Phi + \mathfrak{A})$. Si æquatio algebraica duos hujusmodi factores habeat æquales, ita ut ſit diviſibilis per $(ff - 2fz\cos A\Phi + zz)^2$ tum ex diviſore quadrato oritur ſequens valor $y = \alpha e^{fx\cos A.\Phi}\sin A(fx\sin A\Phi + \mathfrak{A}) + \beta x e^{fx\cos A\Phi}\sin A.(fx\sin A\Phi + \mathfrak{B})$. Sin autem hujus factoris poteſtas quæcunque puta $(ff - 2fz\cos A\Phi + zz)^k$ fuerit diviſor æquationis algebraicæ, tum ex eo reſultat ſequens valor $y =$

$$\alpha e^{fx\cos A\Phi}\sin A(fx\sin A\Phi + \mathfrak{A}) + \beta x e^{fx\cos A\Phi}\sin A(fx\sin A\Phi + \mathfrak{B}) + \gamma x^2 e^{fx\cos A\Phi}\sin A(fx\sin A\Phi + \mathfrak{C}) + \delta x^3 e^{fx\cos A\Phi}\sin A(fx\sin A\Phi + \mathfrak{D}) + \ldots\ldots + \varkappa x^{k-1} e^{fx\cos A\Phi}\sin A(fx\sin A\Phi + \mathfrak{K})$$

Inventis autem hoc modo ex ſingulis diviſoribus æquationis algebraicæ convenientibus valoribus ipſius y, nihil aliud ſupereſt, niſi ut omnes iſti valores in unam ſummam collocentur, hocque modo prodibit valor ipſius y completus; atque is ipſe, qui proditurus eſſet, ſi æquatio differentialis gradus n propoſita n vicibus integraretur. Q. E. J.

Exemplum. I.

§ XXIX. *Hujus æquationis differentialis ſecundi gradus* $o = ay + \frac{bdy}{dx} + \frac{cddy}{dx^2}$ *integrale invenire:*

Poſitis uti præcepimus 1 pro y; z pro $\frac{dy}{dx}$, & zz pro $\frac{ddy}{dx^2}$, oritur hæc æquatio $o = a + bz + czz$; quæ vel ambas radices habebit reales, vel imaginarias: prius evenit, ſi $bb > 4ac$, poſterius ſi $bb < 4ac$. Sit igitur primo $bb > 4ac$, ac duæ radices erunt $z = \frac{-b \pm \sqrt{(bb-4ac)}}{2c}$ hæque caſu erit integrale quæſitum.

$$y = \alpha e^{\frac{-bx + x\sqrt{(bb-4ac)}}{2c}} + \beta e^{\frac{-bx - x\sqrt{(bb-4ac)}}{2c}}$$

Caſus hic ſeorſim eſt perpendendus, quo $bb = 4ac$, tum enim erit $a + 2z\sqrt{ac} + czz$ quadratum nempe $(\sqrt{a} + z\sqrt{c})^2$, quod comparatum cum forma $(f-z)^2$ dat $f = -\sqrt{\frac{a}{c}}$, ex quo integrale erit $y = (\alpha + \beta x)e^{-x\sqrt{\frac{a}{c}}}$, quod eſt integrale æquationis $o = ay + \frac{2dy\sqrt{ac}}{dx} + \frac{cddy}{dx^2}$.

Sit jam $bb < 4ac$, atque æquatio $o = a + bz + czz$ non habebit radices reales, comparata ergo cum forma $ff - 2fz\,\mathrm{coſ}\,A\Phi + zz$ dat $\frac{b}{c} = -2f\,\mathrm{coſ}\,A\Phi$, & $\frac{a}{c} = ff$; unde

de erit $f = V\frac{a}{c}$ & coſ A $\Phi = \frac{-b}{2V ac}$ hincque ſin A $\Phi =$

$\frac{V(4ac-bb)}{2V ac}$, ex quo oritur integrale $y = \alpha\, e^{\frac{-bx}{2c}}$ ſin A $\left(\frac{x V(4ac-bb)}{2c} + \mathfrak{A}\right)$.

Exemplum. 2.

§. XXX. *Hujus æquationis differentialis tertii gradus* $0 = y - \frac{3a^2 ddy}{dx^2} + \frac{2a^3 d^3 y}{dx^3}$ *integrale invenire.*

Ex hac æquatione ergo oritur iſta algebraica $0 = 1 - 3a^2 zz + 2a^3 z^3$, quæ reſolvitur in hos factores $(1+2az)(1-az)^2$. Prior factor $1+2az$ cum forma $f-z$ comparatus dat $f = \frac{-1}{2a}$ unde oritur $y = \alpha\, e^{\frac{-x}{2a}}$; Poſterior factor $(1-az)^2$ comparari debet cum cum $(f-z)^2$, ex quo fit $f = \frac{1}{a}$ hincque nascitur $y = (\beta + \gamma x)\, e^{\frac{x}{a}}$. Æquationis ergo propoſitæ integrale completum erit

$$y = \alpha\, e^{\frac{-x}{2a}} + (\beta + \gamma x)\, e^{\frac{x}{a}}.$$

Exemplum. 3.

§. XXXI. *Hujus æquationis differentialis tertii gradus* $0 = y - \frac{a^3 d^3 y}{dx^3}$ *integrale invenire.*

Æquatio

Æquatio algebraica ex hac æquatione orta erit $0 = 1 - a^3 z^3$, quæ reſolvitur in hos factores: $(1-az)(1+az+a^2zz)$ ita ut ejus diviſores ſint hi $\frac{1}{a} - z$ & $\frac{1}{aa} + \frac{z}{a} + zz$ quorum iſte in ſimplices reales reſolvi nequit. Ille igitur diviſor $\frac{1}{a} - z$ dat pro integrali $y = \alpha\, e^{\frac{x}{a}}$; alter vero diviſor $\frac{1}{aa} + \frac{z}{a} + zz$ cum forma $ff - 2fz \operatorname{coſ} A\Phi + zz$ comparatus dat $f = \frac{1}{a}$ & $\frac{-2 \operatorname{coſ} A\Phi}{a} = \frac{1}{a}$, ita ut fiat $\operatorname{coſ} A\Phi = \frac{-1}{2}$ & $\operatorname{ſin} A\Phi = \frac{\sqrt{3}}{2}$ hinc ex iſto diviſore reſultat $y = \beta\, e^{\frac{-x}{2a}} \operatorname{ſin} A \left(\frac{x\sqrt{3}}{2a} + \mathfrak{A}\right)$. Æquationis ergo propoſitæ integrale completum erit

$$y = \alpha\, e^{\frac{x}{a}} + \beta\, e^{\frac{-x}{2a}} \operatorname{ſin} A \left(\frac{x\sqrt{3}}{2a} + \mathfrak{A}\right).$$

Exemplum. 4.

§. XXXII. *Hujus æquationis differentialis quarti gradus* $0 = y - \frac{a^4 d^4 y}{dx^4}$ *integrale invenire.*

Ex hac æquatione formabitur iſta æquatio algebraica $0 = 1 - a^4 z^4$, quæ duos habet diviſores ſimplices reales $\frac{1}{a} - z$ & $\frac{1}{a} + z$, reliqui duo imaginarii continentur in

hoc

hoc compoſito $\frac{1}{aa} + zz$. Bini diviſores ſimplices pro integrali dant $y = \alpha e^{\frac{x}{a}} + \beta e^{\frac{-x}{a}}$. Diviſor autem $\frac{1}{aa} + zz$ cum forma $ff - 2fz \cos A\Phi + zz$ comparatus dat $f = \frac{1}{a}$, & $\cos A\Phi = 0$, hincque $\sin A\Phi = 1$: Terminus ergo exponentialis $e^{fx \cos A\Phi}$ ob exponentem $= 0$ abit in unitatem, eritque $y = \gamma \sin A \left(\frac{x}{a} + \mathfrak{A}\right)$. Integrale ergo completum erit:

$$y = \alpha e^{\frac{x}{a}} + \beta e^{\frac{-x}{a}} + \gamma \sin A\left(\frac{x}{a} + \mathfrak{A}\right)$$

Exemplum. 5.

§. XXXIII. *Hujus æquationis differentialis quarti gradus* $0 = y + \frac{a^4 d^4 y}{dx^4}$ *integrale invenire.*

Reſolvi ergo oportebit iſtam æquationem algebraicam $0 = 1 + a^4 z^4$, quæ cum nullum habeat diviſorem ſimplicem realem, reſolvitur in hos binos factores compoſitos reales $1 + az\sqrt{2} + aazz$ & $1 - az\sqrt{2} + aazz$, qui diviſi per aa, ut cum forma $ff - 2fz \cos A\Phi + zz$ comparari queant, dabunt $\frac{1}{aa} + \frac{z\sqrt{2}}{a} + zz$ & $\frac{1}{aa} - \frac{z\sqrt{2}}{a} + zz$

Ff pro

pro utraque fit $f = \frac{1}{a}$; pro illa autem, f coſ A Φ $= \frac{-1}{a\sqrt{2}}$, at pro hac f coſ A $\Phi = \frac{1}{a\sqrt{2}}$; hincque iterum pro utraque f ſin A $\Phi = \frac{1}{a\sqrt{2}}$. Ex quibus oritur integrale completum æquationis propoſitæ

$$y = \alpha\, e^{\frac{-x}{a\sqrt{2}}} \text{ſin A} \left(\frac{x}{a\sqrt{2}} + \mathfrak{A}\right) + \beta e^{\frac{x}{a\sqrt{2}}} \text{ſin A} \left(\frac{x}{a\sqrt{2}} + \mathfrak{B}\right)$$

Exemplum. 6.

§. XXXIV. *Hujus æquationis differentialis ſeptimi gradus* $0 = y + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \frac{d^4y}{dx^4} + \frac{d^5y}{dx^5} + \frac{d^7y}{dx^7}$ *integrale completum invenire.*

Naſcitur hinc iſta æquatio algebraica 7^{mi} ordinis

$$0 = 1 + zz + z^3 + z^4 + z^5 + z^7$$

quæ reſolvitur in ſequentes factores reales tam ſimplices quam compoſitos: $(1+z)$ $(1+z+zz)$ $(1-z+zz)^2$. Horum primus cum forma $f - z$ comparatus dat $f = -1$, hincque oritur $y = \alpha e^{-x}$. Factor autem alter $1+z+zz$ comparatus cum $ff - 2fz$ coſ A $\Phi + zz$ dat $f = 1$ & coſ A $\Phi = -\frac{1}{2}$, unde ſin A $\Phi = \frac{\sqrt{3}}{2}$, & integrale hinc natum $y = \beta\, e^{\frac{-x}{2}} \text{ſin A} \left(\frac{x\sqrt{3}}{2} + \mathfrak{A}\right)$. Tertius factor $(1-z+zz)^2$

compa-

comparari debet cum forma $(ff-2fz\cos \mathrm{A}\Phi+zz)^2$, unde fit $f=1$, $\cos \mathrm{A}\Phi=\frac{1}{2}$ & $\sin \mathrm{A}\Phi=\frac{\sqrt{3}}{2}$. Ex eo igitur prodit integrale $y=\gamma e^{\frac{x}{2}}\sin \mathrm{A}\left(\frac{x\sqrt{3}}{2}+\mathfrak{B}\right)+\delta x e^{\frac{x}{2}}\sin \mathrm{A}\left(\frac{x\sqrt{3}}{2}+\mathfrak{C}\right)$. Quamobrem æquationis differentialis propoſitæ integrale completum erit

$$y=\alpha e^{-x}+\beta e^{\frac{-x}{2}}\sin \mathrm{A}\left(\frac{x\sqrt{3}}{2}+\mathfrak{A}\right)+\gamma e^{\frac{x}{2}}\sin \mathrm{A}\left(\frac{x\sqrt{3}}{2}+\mathfrak{B}\right)+\delta x e^{\frac{x}{2}}\sin \mathrm{A}\left(\frac{x\sqrt{3}}{2}+\mathfrak{C}\right)$$

in quo utique ſeptem conſtantes arbitrariæ continentur.

Exemplum. 7.

§. XXXV. *Hujus æquationis differentialis octavi gradus*

$$0=\frac{d^3y}{dx^3}-\frac{3d^4y}{dx^4}+\frac{4d^5y}{dx^5}-\frac{4d^6y}{dx^6}+\frac{3d^7y}{dx^7}-\frac{d^8y}{dx^8}$$

integrale completum invenire.

Æquatio algebraica octavi gradus, quam reſolvi oportet, erit $0=z^3-3z^4+4z^5-4z^6+3z^7-z^8$: quam primum diviſibilem eſſe conſtat per z^3: qui diviſor cum forma $(f-z)^3$ comparatus dat $f=0$, hincque pro integrali invenitur $y=\alpha+\beta x+\gamma xx$. Diviſore hoc in computum ducto ſupereſt reſolvenda hæc æquatio $0=1$

 $-3z$

$-3z+4zz-4z^3+3z^4-z^5$, quæ per $1+zz$ diviſibilis deprehenditur, quo cum forma $ff-2fz\cos A\Phi+zz$ comparato fit $f=1$ & $\cos A\Phi=0$ unde $\sin A\Phi=1$; ideoque reſultat $y=\delta\sin A\,(x+\mathfrak{A})$. Diviſione porro per $1+zz$ inſtituta remanet æquatio $1-3z+3zz-z^3=0=(1-z)^3$; in forma ergo $(f-z)^3$ fit $f=1$, atque integrale hinc oriundum eſt $y=(\varepsilon+\zeta x+\eta xx)e^x$. Conſequenter completum integrale æquationis propoſitæ eſt $y=\alpha+\beta x+\gamma xx+\delta\sin A\,(x+\mathfrak{A})+(\varepsilon+\zeta x+\eta xx)e^x$.

Exemplum. 8.

§. XXXVI. *Hujus æquationis differentialis indefiniti gradus* $0=\frac{d^n y}{dx^n}$ *integrale invenire.*

Hinc reſultat iſta æquatio algebraica $z^n=0$, cujus cum omnes radices ſint æquales, ea comparari debet cum factore $(f-z)^k$, fietque $k=n$ & $f=0$ ex quo ſtatim prodit integrale quæſitum

$$y=\alpha+\beta x+\gamma x^2+\delta x^3+\ldots\ldots+\nu x^{n-1}$$

Hoc vero idem integrale facile invenitur integrationem n vicibus ſucceſſive inſtituendo. Prima enim integratione oritur $\alpha=\frac{d^{n-1}y}{dx^{n-1}}$; multiplicetur per dx & integretur ſecundo erit $\alpha x+\beta=\frac{d^{n-2}y}{dx^{n-2}}$. Quæ per dx multiplicata &

tertio

tertio integrata dabit $\frac{\alpha x x}{2} + \beta x + \gamma = \frac{d^{n-3} y}{d x^{n-3}}$. Atque ita porro, ſi integratio n vicibus repetatur, prodibit mutatis conſtantium expreſſionibus id ipſum integrale, quod per noſtram regulam invenimus.

§. XXXVII.

Hujus methodi beneficio poſſunt etiam plurimæ aliæ æquationes differentialis gradus indefiniti integrari, quæ quidem ad æquationes algebraicas deducunt, quarum factores reales ſive ſimplices ſive compoſiti actu exhiberi poſſunt. Cum autem hujus loci non ſit modum tradere, diviſores hujusmodi æquationum indefiniti dimenſionum numeri inveſtigandi, hic ejusmodi æquationes differentiales inſuper tractabimus, quæ ad æquationes algebraicas perducunt, quarum factores jam aliunde ſunt cogniti. Hujusmodi æquationes autem ſunt $f^n \pm z^n = 0$, & $f^{2n} \pm 2 p f^n z^n \pm z^{2n} = 0$; harum enim expreſſionum factores reales tam ſimplices quam compoſiti ſeu trinomiales omnes exhibiti ſunt a Viris de Analyſi meritiſſimis Cotesio & Moivræo, quos proinde tanquam cognitos in ſolutione ſequentium problematum asſumemus.

Problema. II.

§. XXVVIII.

Si propoſita fuerit iſta æquatio differentialis gradus n;

$0 = y - \frac{d^n y}{dx^n}$, in qua elementum dx ponitur conſtans, ejus integrale completum invenire.

Solutio.

Poſito uti præſcripſimus 1 loco y & z^n loco $\frac{d^n y}{dx^n}$ habebitur iſta æquatio algebraica $0 = 1 - z^n$, cujus æquationis diviſor ſimplex perpetuo $1 - z$, &, ſi n fuerit numerus par, diviſor ſimplex quoque eſt $1 + z$. Reliqui diviſores ſimplices omnes ſunt imaginarii, iique continentur in hac forma generali $1 - 2z \operatorname{cof} A. \frac{2k\pi}{n} + zz$; (ubi π denotat ſemicircumferentiam circuli cujus radius $= 1$) qui cum diviſore trinomiali generali $ff - 2fz \operatorname{cof} A \Phi + zz$ comparatus dat $f = 1$ & $\Phi = \frac{2k\pi}{n}$, ita ut hic diviſor det valorem integralem $y = \alpha\, e^{x \operatorname{cof} A \frac{2k\pi}{n}} \operatorname{fin} A \left(x \operatorname{fin} A \frac{2k\pi}{n} + \mathfrak{A}\right)$ Quodſi jam loco $2k$ ſucceſſive omnes numeri pares exponentem n non excedentes ſubſtituantur, prodibunt omnes poſſibiles valores, qui pro y ſubſtituti ſatisfaciunt. Continetur vero etiam in hac generali forma valor ipſius y, qui oritur ex factore ſimplici $1 - z$ qui eſt $y = \alpha\, e^{x}$; poſito enim $k = 0$, fit $\operatorname{cof} A. \frac{2k\pi}{n} = 1$. & $\operatorname{fin} A. \frac{2k\pi}{n} = 0$, hincque y

$=$

$= \alpha e^{x}$, ob ſin A. $\mathfrak{A}$ conſtantem in α complexum. Simili modo, ſi n ſit numerus par, valor ipſius y ex factore $1 + z$ oriundus, qui eſt $y = \alpha e^{-x}$, ex factore generali reſultat facto $2k = n$, fit enim tum cof A. $\frac{2k\pi}{n} = -1$ & ſin A. $\frac{2k\pi}{n}$ $= 0$, ita ut valor ex factore generali oriundus $y = \alpha e^{-x}$. Integrale ergo completum obtinebitur, ſi in forma generali $y = \alpha e^{x \operatorname{cof} A. \frac{2k\pi}{n}} \operatorname{ſin} A (x \operatorname{ſin} A. \frac{2k\pi}{n} + \mathfrak{A})$ ſucceſſive loco $2k$ omnes numeri pares a o usque ad n ſubſtituantur, hique valores in unam ſummam conjiciantur: Prodibit ergo integrale quæſitum & completum

$$y = \alpha e^{x} + \beta e^{x \operatorname{cof} A \frac{2\pi}{n}} \operatorname{ſin} A (x \operatorname{ſin} A. \tfrac{2\pi}{n} + \mathfrak{B})$$
$$+ \gamma e^{x \operatorname{cof} A. \frac{4\pi}{n}} \operatorname{ſin} A (x \operatorname{ſin} A \tfrac{4\pi}{n} + \mathfrak{C})$$
$$+ \delta e^{x \operatorname{cof} A. \frac{6\pi}{n}} \operatorname{ſin} A (x \operatorname{ſin} A. \tfrac{6\pi}{n} + \mathfrak{D})$$
$$+ \varepsilon e^{x \operatorname{cof} A. \frac{8\pi}{n}} \operatorname{ſin} A (x \operatorname{ſin} A. \tfrac{8\pi}{n} + \mathfrak{E}) + \&c.$$

quæ membra eousque debent continuari, quoad n conſtantes arbitrariæ habeantur, vel quod eodem redit, quoad coefficiens ipſius π major unitate evadat. Fiet autem, ſi n ſit nume-

numerus impar ultimum membrum $= \nu\, e^{x \cos A. \frac{(n-1)\pi}{n}}$ $\sin A\,(x \sin A. \frac{(n-1)\pi}{n} + \mathfrak{N})$; at ſi n ſit numerus par, erit ultimum membrum $= \nu\, e^{-x}$; & penultimum $= \mu\, e^{x \cos A \frac{(n-2)\pi}{n}} \sin A\,(x \sin A \frac{(n-2)\pi}{n} + \mathfrak{M}.)$ Pro quovis ergo valore ipſius n integrale completum expedite asſignabitur. Q. E. J.

§. XXXIX.

Quo iſta integralia clarius ob oculos ponantur pro ſingulis valoribus ipſius n ab unitate incipiendo integralia æquationis $0 = y - \frac{d^n y}{dx^n}$ exhibeamus:

I. Hujus æquationis $0 = y - \frac{dy}{dx}$ integrale eſt:

$$y = \alpha\, e^{x}$$

II. Hujus æquationis $0 = y - \frac{ddy}{dx^2}$ integrale eſt:

$$y = \alpha\, e^{x} + \beta\, e^{-x}$$

III. Hujus æquationis $0 = y - \frac{d^3 y}{dx^3}$ integrale eſt:

$$y = \alpha\, e^{x} + \beta\, e^{x \cos A \frac{2}{3}\pi} \sin A\,(x \sin A. \tfrac{2}{3}\pi + \mathfrak{B})$$

IV. Hujus æquationis $0 = y - \frac{d^4 y}{dx^4}$ integrale eſt:

$y =$

$$y = \alpha e^{x} + \beta \,\mathrm{ſin\,A}\,(x + \mathfrak{B}) + \gamma e^{-x}$$

V. Hujus æquationis $o = y - \frac{d^5 y}{dx^5}$ integrale eſt:

$$y = \alpha e^{x} + \beta e^{x \,\mathrm{coſ\,A}\frac{2}{5}\pi} \,\mathrm{ſin\,A}(x\,\mathrm{ſin\,A}.\tfrac{2}{5}\pi + \mathfrak{B}) + \gamma e^{x\,\mathrm{coſ\,A}\frac{4}{5}\pi} \,\mathrm{ſin\,A}(x\,\mathrm{ſin\,A}\tfrac{4}{5}\pi + \mathfrak{C})$$

VI. Hujus æquationis $o = y - \frac{d^6 y}{dx^6}$ integrale eſt:

$$y = \alpha e^{x} + \beta e^{x\,\mathrm{coſ\,A}\frac{1}{3}\pi} \,\mathrm{ſin\,A}\,(x\,\mathrm{ſin\,A}.\tfrac{1}{3}\pi + \mathfrak{B}) + \gamma e^{x\,\mathrm{coſ\,A}\frac{2}{3}\pi} \,\mathrm{ſin\,A}\,(x\,\mathrm{ſin\,A}.\tfrac{2}{3}\pi + \mathfrak{C}) + \delta e^{-x}$$

VII. Hujus æquationis $o = y - \frac{d^7 y}{dx^7}$ integrale eſt.

$$y = \alpha e^{x} + \beta e^{x\,\mathrm{coſ\,A}\frac{2}{7}\pi} \,\mathrm{ſin\,A}\,(x\,\mathrm{ſin\,A}\tfrac{2}{7}\pi + \mathfrak{B}) + \gamma e^{x\,\mathrm{coſ\,A}\frac{4}{7}\pi} \,\mathrm{ſin\,A}\,(x\,\mathrm{ſin\,A}\tfrac{4}{7}\pi + \mathfrak{C}) + \delta e^{x\,\mathrm{coſ\,A}\frac{6}{7}\pi} \,\mathrm{ſin\,A}\,(x\,\mathrm{ſin\,A}\tfrac{6}{7}\pi + \mathfrak{D})$$

VIII. Hujus æquationis $o = y - \frac{d^8 y}{dx^8}$ integrale eſt:

$$y = \alpha e^{x} + \beta e^{x\,\mathrm{coſ\,A}\frac{1}{4}\pi} \,\mathrm{ſin\,A}\,(x\,\mathrm{ſin\,A}\tfrac{1}{4}\pi + \mathfrak{B}) + \gamma \,\mathrm{ſin\,A}(x + \mathfrak{C}) + \delta e^{x\,\mathrm{coſ\,A}\frac{3}{4}\pi} \,\mathrm{ſin\,A}\,(x\,\mathrm{ſin\,A}\tfrac{3}{4}\pi + \mathfrak{D}) + \varepsilon e^{-x} \quad \&c.$$

Problema. III.

§. XL. Si propoſita fuerit iſta æquatio differentialis gradus indefiniti n; $o = y + \frac{d^n y}{dx^n}$, poſito elemento dx conſtante, ejus integrale invenire.

Solutio.

Poſito ſecundum regulam 1 pro y, & z^n pro $\frac{d^n y}{dx^n}$, prodibit iſta æquatio algebraica $0 = 1 + z^n$, quæ ſi n fuerit numerus impar, diviſorem ſimplicem realem habet $1 + z$, ex quo oritur $y = \alpha e^{-x}$. Reliqui diviſores ſimplices omnes ſunt imaginarii; horum vero bini continentur in hoc factore trinomiali reali: $1 - 2z \operatorname{coſ A}. \frac{2k-1}{n}\pi + zz$, hæcque expreſſio omnes prorſus diviſores formæ $1 + z^n$ ſuggerit, ſi loco $2k-1$ omnes numeri impares ipſo n non majores ſucceſſive ſubſtituantur. Collata autem hac formula $1 - 2z \operatorname{coſ A}. \frac{2k-1}{n}\pi + zz$ cum factore generali $ff - 2fz \operatorname{coſ A} \Phi + zz$ fit $f = 1$ & $\Phi = \frac{2k-1}{n}\pi$; hinc ergo enaſcitur ſequens pro y valor generalis $y = \alpha e^{x \operatorname{coſ A} \frac{2k-1}{n}\pi} \operatorname{ſin A} \left(x \operatorname{ſin A} \frac{2k-1}{n}\pi + \mathfrak{A}\right)$. Atque in hoc valore generali etiam continetur valor ipſius y ex factore ſimplici $1 + z$, ſi quidem n fuerit numerus impar, oriundus; prodit enim hoc caſu $y = \alpha e^{-x}$, ſi fiat $2k-1 = n$, tum enim fit $\operatorname{coſ A} \frac{2k-1}{n}\pi = \operatorname{coſ A}\pi = -1$, ejusque ſinus $= 0$. Integrale ergo completum æquationis propoſitæ invenietur, ſi in hac forma generali $y =$

$\alpha\ e$

$$\alpha e^{x \cos A \frac{2k-1}{n}\pi} \sin A \left(x \sin A \frac{2k-1}{n} \pi + \mathfrak{A}\right)$$ loco $2k-1$

ſucceſſive omnes numeri impares 1, 3, 5, 7, &c. qui quidem exponente n non ſunt majores, ſubſtituantur, iſtique valores cuncti in unam ſummam colligantur. Prodibit ergo hoc modo integrale quæſitum & completum:

$$y = \alpha e^{x \cos A \frac{1}{n}\pi} \sin A \left(x \sin A. \frac{1}{n} \pi + \mathfrak{A}\right)$$

$$+ \beta e^{x \cos A. \frac{3}{n}\pi} \sin A \left(x \sin A. \frac{3}{n} \pi + \mathfrak{B}\right)$$

$$+ \gamma e^{x \cos A. \frac{5}{n}\pi} \sin A \left(x \sin A. \frac{5}{n} \pi + \mathfrak{C}\right)$$

$$+ \delta e^{x \cos A. \frac{7}{n}\pi} \sin A \left(x \sin A \frac{7}{n} \pi + \mathfrak{D}\right) + \&c.$$

quæ membra eousque continuari debent, quoad n conſtantes arbitrariæ fuerint ingreſſæ; quod eveniet, ſi ex ſerie fractionum $\frac{1}{n}$, $\frac{3}{n}$, $\frac{5}{n}$, $\frac{7}{n}$ &c. omnes unitatem non ſuperantes capiantur. Fiet autem, ſi n ſit numerus par, membrum ultimum $\nu e^{x \cos A \frac{n-1}{n}\pi} \sin A \left(x \sin A \frac{n-1}{n} \pi + \mathfrak{N}\right)$.

At

At ſi n numerus impar, membrum ultimum erit: $\nu\, e^{-x}$, pe-
nultimum vero $\mu\, e^{x \operatorname{coſ} A \frac{n-2}{n}\pi} \operatorname{ſin} A (x \operatorname{ſin} A \frac{n-2}{n}\pi + \mathfrak{M})$
unde nullo negotio integrale completum quovis caſu aſſignatur. Q. E. J.

§. XLI.

Ad integrale hoc diſtinctius explicandum aliquot caſus ſimpliciores evolvamus, loco n ponendo, ut in præcedente problemate fecimus, numeros integros 1, 2, 3 . . . 8; quo vis hujus integrationis penitius perſpiciatur.

I. Hujus æquationis $0 = y + \frac{dy}{dx}$ integrale eſt:

$$y = \alpha\, e^{-x}$$

II. Hujus æquationis $0 = y + \frac{ddy}{dx^2}$ integrale eſt:

$$y = \alpha \operatorname{ſin} A\ (x + \mathfrak{A})$$

III. Hujus æquationis $0 = y + \frac{d^3y}{dx^3}$ integrale eſt:

$$y = \alpha\, e^{x \operatorname{coſ} A \frac{1}{3}\pi} \operatorname{ſin} A (x \operatorname{ſin} A \tfrac{1}{3}\pi + \mathfrak{A}) + \beta\, e^{-x}$$

IV. Hujus æquationis $0 = y + \frac{d^4y}{dx^4}$ integrale eſt:

$$y = \alpha\, e^{x \operatorname{coſ} A \frac{1}{4}\pi} \operatorname{ſin} A (x \operatorname{ſin} A. \tfrac{1}{4}\pi + \mathfrak{A}) + \beta\, e^{x \operatorname{coſ} A \frac{3}{4}\pi} \operatorname{ſin} A (x \operatorname{ſin} \tfrac{3}{4}\pi + \mathfrak{B})$$

V. Hujus æquationis $0 = y + \frac{d^5y}{dx^5}$ integrale eſt:

$y =$

$$y = \alpha e^{x\cos A\frac{1}{5}\pi} \sin A\,(x \sin A\,\tfrac{1}{5}\pi + \mathfrak{A}) + \beta e^{x\cos A\frac{3}{5}\pi} \sin A\,(x \sin A.\tfrac{3}{5}\pi + \mathfrak{B}) + \gamma e^{-x}$$

VI. Hujus æquationis $0 = y + \frac{d^6 y}{dx^6}$ integrale eſt:

$$y = \alpha e^{x\cos A\frac{1}{6}\pi} \sin A\,(x \sin A\,\tfrac{1}{6}\pi + \mathfrak{A}) + \beta \sin A\,(x + \mathfrak{B}) + \gamma e^{x\cos A\frac{5}{6}\pi} \sin A\,(x \sin A\,\tfrac{5}{6}\pi + \mathfrak{C})$$

VII. Hujus æquationis $= y + \frac{d^7 y}{dx^7}$ integrale eſt:

$$y = \alpha e^{x\cos A\frac{1}{7}\pi} \sin A\,(x \sin A.\tfrac{1}{7}\pi + \mathfrak{A}) + \beta e^{x\cos A.\frac{3}{7}\pi} \sin A.\,(x \sin A\,\tfrac{3}{7}\pi + \mathfrak{B}) + \gamma e^{x\cos A.\frac{5}{7}\pi} \sin A\,(x \sin A\,\tfrac{5}{7}\pi + \mathfrak{C}) + \delta e^{-x}$$

VIII. Hujus æquationis: $0 = y + \frac{d^8 y}{dx^8}$ integrale eſt:

$$y = \alpha e^{x\cos A\frac{1}{8}\pi} \sin A(x \sin A\,\tfrac{1}{8}\pi + \mathfrak{A}) + \beta e^{x\cos A.\frac{3}{8}\pi} \sin A(x \sin A.\tfrac{3}{8}\pi + \mathfrak{B}) + \gamma e^{x\cos A\frac{5}{8}\pi} \sin A(x \sin A.\tfrac{5}{8}\pi + \mathfrak{C}) + \delta e^{x\cos A\frac{7}{8}\pi} \sin A\,(x \sin A.\tfrac{7}{8}\pi + \mathfrak{D})$$

Problema. IV.

§. XLII.

Si propoſita fuerit æquatio differentialis gradus $2n$ hæc: $0 = y + \frac{2b\,d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$ poſito elemento dx conſtante; ejus integrale invenire; exiſtente $bb > 1$.

 Solu-

Si ſecundum regulam ponamus 1 pro y; z^n pro $\frac{d^n y}{dx^n}$, & z^{2n} pro $\frac{d^{2n} y}{dx^{2n}}$, prodit iſta æquatio algebraica $0 = 1 + 2b\, z^n + z^{2n}$, quæ ob $bb > 1$ in hos duos factores reſolvitur:

$$(z^n + b + \sqrt{(bb-1)})\,(z^n + b - \sqrt{(bb-1)})$$

Ponimus autem hic b quantitatem affirmativam; & hanc obrem tam $b + \sqrt{(bb-1)}$ quam $b - \sqrt{(bb-1)}$ erunt quantitates affirmativæ. Sit ergo $b + \sqrt{(bb-1)} = a^n$ & $b - \sqrt{(bb-1)} = b^n$, ita ut ſit $ab = 1$. Habebimus igitur iſtam æquationem in duos factores reales reſolutam: $0 = (z^n + a^n)\,(z^n + b^n)$: atque prioris factoris ſinguli factores trinomiales reales continebuntur in hac forma: $aa - 2az\, \text{coſA}\, \frac{2k-1}{n}\pi + zz$: poſterioris vero in hac: $bb - 2bz\, \text{coſA}\, \frac{2k-1}{n}\pi + zz$: Omneſque factores habebuntur, ſi in utraque forma ſucceſſive pro $2k-1$ ponantur omnes numeri impares 1, 3, 5, 7, &c. qui exponente n non ſint majores. Integrale ergo quæſitum ex his factoribus ita formabitur:

$$y = \mathrm{A}e^{ax\,\text{coſA}\frac{1}{n}\pi}\,\text{ſinA}\left(ax\,\text{ſinA}\tfrac{1}{n}\pi + \mathfrak{A}\right) + \mathrm{B}e^{ax\,\text{coſA}\frac{3}{n}}\,\text{ſinA}\left(ax\,\text{ſinA}\tfrac{3}{n}\pi + \mathfrak{B}\right)$$

$$+ \mathrm{C}e^{ax\,\text{coſA}\frac{5}{n}\pi}\,\text{ſinA}\left(ax\,\text{ſinA}\tfrac{5}{n}\pi + \mathfrak{C}\right) + \mathrm{D}e^{ax\,\text{coſA}\frac{7}{n}\pi}\,\text{ſinA}\left(ax\,\text{ſinA}\tfrac{7}{n}\pi + \mathfrak{D}\right)$$

$$+ \text{\&c.} \qquad + a$$

$$+\alpha e^{bx\cos A\frac{1}{n}\pi}\sin A\left(bx\sin A\tfrac{1}{n}\pi+\mathfrak{a}\right)+\beta e^{bx\cos A\frac{3}{n}\pi}\sin A\left(bx\sin A\tfrac{3}{n}\pi+\mathfrak{b}\right)$$

$$+\gamma e^{bx\cos A\frac{5}{n}\pi}\sin A\left(bx\sin A\tfrac{5}{n}\pi+\mathfrak{c}\right)+\&c.$$ Q. E. J.

Problema. V.

§. XLIII.

Si propoſita fuerit æquatio differentialis gradus indefiniti $2n$ hæc: $0 = y - \frac{2bd^ny}{dx^n} + \frac{d^{2n}y}{dx^{2n}}$ ſumto dx conſtante, & exiſtente $bb \succeq 1$; ejus integrale invenire.

Solutio.

Secundum regulam ſupra datam orietur hinc ſequens æquatio algebraica $0 = 1 - 2bz^n + z^{2n}$; quæ in hos duos factores reales primum reſolvitur:

$$0 = (z^n - b + \sqrt{(bb-1)})\ (z^n - b - \sqrt{(bb-1)}.)$$

Cum autem b denotet quantitatem poſitivam, ponatur $b + \sqrt{(bb-1)} = a^n$, & $b - \sqrt{(bb-1)} = b^n$, ita ut ſit $ab = 1$; hincque orietur iſta æquatio:

$$0 = (z^n - a^n)\ (z^n - b^n)$$

Prioris factoris $z^n - a^n$ omnes factores trinomiales reales continentur in hac forma $aa - 2az\cos A.\frac{2k}{n}\pi + zz$; poſterioris vero $z^n - b^n$ in hac forma $bb - 2bz\cos A.\frac{2k}{n}\pi + zz$:

zz: omneſque factores habebuntur, ſi in utraque forma loco $2k$ ſucceſſive ponantur omnes numeri pares 0, 2, 4, 6, &c. numero n non majores. Ex his itaque factoribus cognitis integrale quæſitum colligitur fore:

$$y = \begin{cases} \mathrm{A}e^{ax} + \mathrm{B}e^{ax\,\text{coſ}\mathrm{A}\frac{2}{n}\pi}\ \text{ſin}\,\mathrm{A}\left(ax\,\text{ſin}\,\mathrm{A}\frac{2}{n}\pi + \mathfrak{B}\right) \\ + \mathrm{C}e^{ax\,\text{coſ}\mathrm{A}.\frac{4}{n}\pi}\ \text{ſin}\,\mathrm{A}\left(ax\,\text{coſ}\mathrm{A}\frac{4}{n}\pi + \mathfrak{C}\right) \\ + \mathrm{D}e^{ax\,\text{coſ}\mathrm{A}.\frac{6}{n}\pi}\ \text{ſin}\,\mathrm{A}\left(ax\,\text{ſin}\,\mathrm{A}\frac{6}{n}\pi + \mathfrak{D}\right) + \&c. \\ \alpha e^{bx} + \beta e^{bx\,\text{coſ}\mathrm{A}\frac{2}{n}\pi}\ \text{ſin}\,\mathrm{A}\left(bx\,\text{ſin}\,\mathrm{A}\frac{2}{n}\pi + \mathfrak{b}\right) \\ + \gamma e^{bx\,\text{coſ}\mathrm{A}.\frac{4}{n}\pi}\ \text{ſin}\,\mathrm{A}\left(bx\,\text{ſin}\,\mathrm{A}\frac{4}{n}\pi + \mathfrak{c}\right) \\ + \delta e^{bx\,\text{coſ}\mathrm{A}\frac{6}{n}\pi}\ \text{ſin}\,\mathrm{A}\left(bx\,\text{ſin}\,\mathrm{A}\frac{6}{n}\pi + \mathfrak{d}\right) + \&c. \end{cases}$$

Q. E. J.

Problema. VI.

§. XLIV. Si propoſita fuerit æquatio differentialis gradus indefiniti hæc: $0 = y + \frac{2b\,d^n y}{dx^n} - \frac{d^{2n} y}{dx^{2n}}$ ſumto elemento dx conſtante; ejus integrale invenire.

So-

Solutio.

Æquatio algebraica, quæ ſecundum præcepta hinc oritur : $0 = 1 + 2bz^n - z^{2n}$ in hos duos factores reales primum reſolvitur:

$$0 = (b + \sqrt{(bb+1)} - z^n)\ (-b + \sqrt{(bb+1)} + z^n)$$

Fiat, id quod ob b quantitatem poſitivam ſemper fieri poteſt $\sqrt{(bb+1)} + b = a^n$, & $\sqrt{(bb+1)} - b = b^n$ ita ut ſit $ab = 1$; hincque naſcetur iſta æquatio:

$$0 = (a^n - z^n)\ (b^n + z^n)$$

Prioris factoris $a^n - z^n$ factores trinomiales reales omnes continentur in hac forma $aa - 2az\,\text{coſ}\,A\frac{2k}{n}\pi + zz$, poſterioris vero in hac: $bb - 2bz\,\text{coſ}\,A\frac{2k-1}{n}\pi + zz$, omneſque factores habebuntur, ſi in priori loco $2k$ omnes numeri pares 0, 2, 4, 6, &c. in poſteriori vero loco $2k-1$ omnes impares 1, 3, 5, 7, &c. numerum n non excedentes ſucceſſive ſubſtituantur. Ex his ergo factoribus cognitis integrale quæſitum colligitur:

$$y = \left\{\begin{array}{l} Ae^{ax} + Be^{ax\,\text{coſ}\,A\frac{2}{n}\pi}\,\text{ſin}\,A\left(ax\,\text{ſin}\,A.\frac{2}{n}\pi + \mathfrak{B}\right) + Ce^{ax\,\text{coſ}\,A\frac{4}{n}\pi} \\ \text{ſin}\,A\left(ax\,\text{ſin}\,A\frac{4}{n}\pi + \mathfrak{C}\right) + De^{ax\,\text{coſ}\,A\frac{6}{n}\pi}\ \text{ſin}\,A\left(ax\,\text{ſin}\,A\frac{6}{n}\pi + \mathfrak{D}\right) \\ \qquad + \&c. \end{array}\right.$$

$$y = \alpha e^{bx \operatorname{cof} A \frac{1}{n}\pi} \operatorname{fin} A.(bx \operatorname{fin} A.\tfrac{1}{n}\pi + \mathfrak{a}) + \beta e^{bx \operatorname{cof} A \frac{3}{n}\pi} \operatorname{fin} A.(bx \operatorname{fin} A \tfrac{3}{n}\pi + \mathfrak{b}) + \gamma e^{bx \operatorname{cof} A. \frac{5}{n}\pi} \operatorname{fin} A.(bx \operatorname{fin} A \tfrac{5}{n}\pi + \mathfrak{c}) + \&c.$$

Q. E. J.

Problema. 7.

§. XLV. Si propoſita fuerit æquatio differentialis gradus indefiniti $2n$ hæc: $0 = y - \frac{2b\,d^n y}{dx^n} - \frac{d^{2n}y}{dx^{2n}}$, in qua poſitum eſt elementum dx conſtans: ejus integrale invenire.

Solutio.

Per ſubſtitutionem per regulam ſupra datam faciendam naſcitur hinc iſta æquatio algebraica ordinis $2n$:

$$0 = 1 - 2bz^n - z^{2n}$$

quæ primum in hos duos factores reales diſcerpitur

$$0 = (-b + \sqrt{(bb+1)} - z^n)\,(b + \sqrt{(bb+1)} + z^n)$$

Ob b quantitatem poſitivam ponatur $\sqrt{(bb+1)} + b = a^n$ & $(bb+1) - b = b^n$; ita ut ſit $ab = 1$. Atque ſequens habebitur æquatio reſolvenda:

$$0 = (a^n + z^n)\,(b^n - z^n)$$

cujus prioris factoris $a^n + z^n$ omnes factores trinomiales continentur in hac forma: $aa - 2az \operatorname{cof} A \frac{2k-1}{n}\pi + zz$ poſterioris

ſterioris vero in hac: $bb - 2bz \cos A.\frac{2k}{n}\pi + zz$; omnesque factores habebuntur, ſi in illa forma ponantur ſucceſſive omnes numeri impares 1, 3, 5, 7, &c. loco $2k-1$; in hac vero loco $2k$ omnes numeri pares 0, 2, 4, 6, &c. numero n non majores. Ex his itaque factoribus colligetur integrale quæſitum & completum : $y =$

$$
\begin{aligned}
& A e^{ax \cos A \frac{1}{n}\pi} \sin A \left(ax \sin A \tfrac{1}{n}\pi + \mathfrak{A}\right) \\
& + B e^{ax \cos A.\frac{3}{n}\pi} \sin A \left(ax \sin A \tfrac{3}{n}\pi + \mathfrak{B}\right) \\
& + C e^{ax \cos A.\frac{5}{n}\pi} \sin A \left(ax \sin A \tfrac{5}{n}\pi + \mathfrak{C}\right) + \&c. \\
& u e^{bx} + \beta e^{bx \cos A \frac{2}{n}\pi} \sin A \left(bx \sin A \tfrac{2}{n}\pi + \mathfrak{b}\right) \\
& + \gamma e^{bx \cos A.\frac{4}{n}\pi} \sin A \left(bx \sin A \tfrac{4}{n}\pi + \mathfrak{c}\right) \\
& + \delta e^{bx \cos A \frac{6}{n}\pi} \sin A \left(bx \sin A \tfrac{6}{n}\pi + \mathfrak{d}\right) + \&c.
\end{aligned}
$$

Q. E. J.

Problema VIII.

§. XLVIII. Si propoſita fuerit hæc æquatio gradus indefiniti: $0 = y + \frac{2b\,d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$, poſito elemento dx conſtante, & exiſtente $bb < 1$. ejus integrale completum invenire.

Solutio.

Æquatio algebraica ordinis $2n$ quæ hinc oritur eſt $0 = 1 + 2bz^n + z^{2n}$, ad cujus factores trinomiales reales omnes inveniendos, capiatur in circulo, cujus radius $= 1$ arcus ω cujus coſinus ſit $= b$, ita ut ſit $b = \operatorname{cof} A\,\omega$. Arcu hoc invento unusquisque factor trinomialis continebitur in hac forma:

$$1 - 2z \operatorname{cof} A \frac{k\pi - \omega}{n} + zz$$

ſubſtituendo loco k omnes numeros impares 1, 3, 5, 7, ... $(2n-1)$; ita ut horum factorum numerus futurus ſit n, uti numerus dimenſionum requirit. Ex his igitur factoribus cognitis reperietur ſecundum præcepta data integrale quæſitum æquationis propoſitæ:

$$y = \alpha e^{x \operatorname{cof} A \frac{\pi-\omega}{n}} \operatorname{fin} A \left(x \operatorname{fin} A \frac{\pi-\omega}{n} + a\right) + \beta e^{x \operatorname{cof} A \frac{3\pi-\omega}{n}} \operatorname{fin} A$$

(x ſin A

$$x \operatorname{coſ} A \frac{5\pi-\omega}{n}$$

$$(x \operatorname{ſin} A \frac{3\pi-\omega}{n} + \mathfrak{b}) + \gamma e \qquad \operatorname{ſin} A (x \operatorname{ſin} A \frac{5\pi-\omega}{n} + \mathfrak{c})$$

$$x \operatorname{coſ} A \frac{(2n-1)\pi-\omega}{n}$$

$$+ \ldots\ldots\ldots + \nu e \qquad \operatorname{ſin} A (x \operatorname{ſin} A \frac{(2n-1)\pi-\omega}{n} + \mathfrak{n})$$

Numerus ſcilicet membrorum hoc integrale conſtituentium eſt n, indeque numerus conſtantium arbitrariarum ingredientium eſt $2n$, uti gradus differentialium æquationis propoſitæ requirit. Q. E. J.

Problema. IX.

§. XLVII. Exiſtente iterum $hh < 1$. ſi propoſita fuerit hæc æquatio differentialis gradus indefiniti $2n$: $0 = y - \frac{2h d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$, ſumto elemento dx conſtante, ejus integrale completum invenire.

Solutio.

Æquatio algebraica, quæ ſecundum præcepta tradita hinc deducitur, eſt $0 = 1 - 2hz^n + z^{2n}$, cujus ſinguli factores trinomiales reales, quorum numerus eſt n continentur in hac forma generali:

$$1 - 2z \operatorname{coſ} A . \frac{h\pi - \omega}{n} + zz$$

 ſi loco

ſi loco k ſucceſſive omnes numeri pares 2, 4, 6, 8, &c. usque ad $2n$ incluſive ſubſtituantur. . Denotat hic autem uti ante ω arcum circuli, cujus coſinus eſt b, qui ob $b < 1$. ſemper aſſignari poteſt, ita ut ſit $b = \cos A\,\omega$. Cognitis autem factoribus omnibus æquationis $0 = 1 - 2bz^n + z^{2n}$, æquationis differentialis propoſitæ integrale completum erit:

$$y = \alpha e^{x \cos A \frac{2\pi-\omega}{n}} \sin A \left(x \sin A . \frac{2\pi-\omega}{n} + \mathfrak{a}\right) + \beta e^{x \cos A \frac{4\pi-\omega}{n}} \sin A \left(x \sin A . \frac{4\pi-\omega}{n} + \mathfrak{b}\right) + \gamma e^{x \cos A . \frac{6\pi-\omega}{n}} \sin A \left(x \sin A . \frac{6\pi-\omega}{n} + \mathfrak{c}\right) + \delta e^{x \cos A . \frac{8\pi-\omega}{n}} \sin A \left(x \sin A \frac{8\pi-\omega}{n} + \mathfrak{d}\right)$$

$$+ \ldots\ldots\ \nu e^{x \cos A . \frac{2n\pi-\omega}{n}} \sin A \left(x \sin A . \frac{2n\pi-\omega}{n} + \mathfrak{n}\right)$$

Ingrediuntur enim in hanc expreſſionem $2n$ conſtantes arbitrariæ. Q. E. J.

Problema. X.

Problema. X.

§. XLVIII. Si propoſita fuerit æquatio differentialis indefiniti gradus hæc: $o = y \pm \frac{2d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$ in qua differentiale dx poſitum eſt conſtans: ejus integrale invenire.

Solutio.

Æquatio algebraica quæ hinc formatur eſt: $o = 1 \pm 2z^n + z^{2n} = (1 \pm z^n)^2$, quæ cum ſit quadratum omnes ejus factores erunt quadrati: pro ſigno ergo ſuperiori hæc forma $\left(1 - 2z \operatorname{coſ} A. \frac{2k-1}{n}\pi + zz\right)^2$ omnes continet factores; pro ſigno inferiori autem hæc forma $\left(1 - 2z \operatorname{coſ} A. \frac{2k}{n}\pi + zz\right)^2$. Ex his factoribus cognitis reperietur pro ſigno inferiori ſeu æquationis $o = y - \frac{2d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}}$ integrale completum:

$$y = \begin{cases} Ae^{x} + Be^{x\operatorname{coſ}A.\frac{2}{n}\pi} \operatorname{ſin}A\left(x\operatorname{ſin}A\frac{2}{n}\pi + \mathfrak{B}\right) + Ce^{x\operatorname{coſ}A.\frac{4}{n}\pi} \\ \operatorname{ſin}A\left(x\operatorname{ſin}A.\frac{4}{n}\pi + \mathfrak{C}\right) + \&c. \\ \alpha x e^{x} + \beta x e^{x\operatorname{coſ}A.\frac{2}{n}\pi} \operatorname{ſin}A\left(x\operatorname{ſin}A\frac{2}{n}\pi + \mathfrak{b}\right) + \gamma x e^{x\operatorname{coſ}A.\frac{4}{n}\pi} \\ \operatorname{ſin}A\left(x\operatorname{ſin}A.\frac{4}{n}\pi + \mathfrak{c}\right) + \&c. \end{cases}$$

Hu-

Hujus vero æquationis $0 = y + \frac{2d^n y}{dx^n} + \frac{d^{2n}y}{dx^{2n}}$ integrale erit

$$y = \begin{cases} A e^{x \,\text{coſ}\, A.\frac{1}{n}\pi} \,\text{ſin}\, A \left(x \,\text{ſin}\, A.\frac{1}{n}\pi + \mathfrak{A}\right) \\ + B e^{x \,\text{coſ}\, A.\frac{3}{n}\pi} \,\text{ſin}\, A \left(x \,\text{ſin}\, A.\frac{3}{n}\pi + \mathfrak{B}\right) \\ + C e^{x \,\text{coſ}\, A.\frac{5}{n}\pi} \,\text{ſin}\, A \left(x \,\text{ſin}\, A.\frac{5}{n}\pi + \mathfrak{C}\right) + \&c. \\ \alpha x e^{x \,\text{coſ}\, A.\frac{1}{n}\pi} \,\text{ſin}\, A \left(x \,\text{ſin}\, A.\frac{1}{n}\pi + \mathfrak{a}\right) \\ + \beta x e^{x \,\text{coſ}\, A.\frac{3}{n}\pi} \,\text{ſin}\, A \left(x \,\text{ſin}\, A.\frac{3}{n}\pi + \mathfrak{b}\right) \\ \gamma x e^{x \,\text{coſ}\, A.\frac{5}{n}\pi} \,\text{ſin}\, A \left(x \,\text{ſin}\, A.\frac{5}{n}\pi + \mathfrak{c}\right) + \&c \end{cases}$$

Q. E. J.

§. XLIX. Ex his allatis exemplis jam abunde perſpicitur quemadmodum omnes æquationes differentiales cujus-

cunque

cunque gradus, quæ quidem contineantur in hac forma

$$o = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \&c.$$

denotantibus litteris A, B, C, D, &c. coefficientes conſtantes, tractari; earum integralia completa inveniri oporteat. Unica nimirum difficultas reſidet in reſolutione æquationum algebraicarum in factores reales ſive ſimplices ſive trinomiales; quam autem in hoc negotio, quippe ab algebra pendentem merito tanquam datam aſſumere poſſumus. At vero hæc eadem methodus adhiberi poteſt quoque in æquationibus hujusmodi, quarum termini in infinitum progrediuntur, dummodo æquationum algebraicarum, quæ inde formantur, omnes aſſignari queant radices. Hunc igitur uſum unico exemplo declarabimus, alia forte occaſione hanc æquationum differentialium infiniti gradus integrationem fuſius expoſituri.

Problema. XI.

§. L. Si propoſita fuerit iſta æquatio differentialis in infinitum excurrens:

$$o = y - \frac{ddy}{2dx^2} + \frac{d^4y}{24dx^4} - \frac{d^6y}{720dx^6} + \frac{d^8y}{40320dx^8} - \&c.$$

in qua differentiale dx poſitum eſt conſtans; ejus integrale completum invenire.

Solutio.

Scripto 1 pro y, & z^k pro differentiali cujusvis gradus

$\frac{d^k y}{dx^k}$, orietur iſta æquatio in infinitum excurrens $o = 1 - \frac{z^2}{1.2.} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.3.4.5.6} + \frac{z^8}{1.2.\ldots.8} -$ &c. quæ convenit cum hac $o = \text{coſ} A. z.$ Hujus ergo æquationis radices ſunt omnes arcus circuli radii $= 1$; quorum coſinus evaneſcunt. Quocirca omnes poſſibiles valores ipſius z erunt ſequentes:

$$\pm \frac{\pi}{2};\quad \pm \frac{3\pi}{2};\quad \pm \frac{5\pi}{2};\quad \pm \frac{7\pi}{2};\quad \pm \frac{9\pi}{2}\quad \&c.$$

His igitur radicibus ac proinde diviſoribus ſimplicibus æquationis illius cognitis, qui omnes ſunt reales; æquationis differentialis propoſitæ integrale completum erit:

$$y = \alpha e^{\frac{\pi x}{2}} + \mathfrak{a} e^{\frac{-\pi x}{x}} + \beta e^{\frac{3\pi x}{2}} + \mathfrak{b} e^{\frac{-3\pi x}{2}} + \gamma e^{\frac{5\pi x}{2}} + \mathfrak{c} e^{\frac{-5\pi x}{2}}$$

$$+ \delta e^{\frac{7\pi x}{2}} + \mathfrak{d} e^{\frac{-7\pi x}{2}} + \&c. \text{ in infinitum.}$$

Atque unus quisque terminus ſeorſim ſumtus, vel plures juncti dabunt integrale particulare æquationis differentialis propoſitæ. Q. E. J.

Con-